1 Definitions

We employ the following definitions for working with equations in cylindrical coordinates (see also: Wikipedia[1]). Note that we use the spelled-out form of the operators (grad , div) to differentiate them from the related operators in Cartesian space.

1. Unit vectors: $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_z$.

2. Components of a vector, $\vec{v}$:

$$\vec{v} = v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \tag{1}$$

3. Components of a tensor, $\boldsymbol{S}$:

$$\boldsymbol{S} = \begin{bmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ S_{\theta r} & S_{\theta\theta} & S_{\theta z} \\ S_{zr} & S_{z\theta} & S_{zz} \end{bmatrix} \tag{2}$$

4. Gradient of a scalar g:

$$\operatorname{grad} g \equiv \frac{\partial g}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial g}{\partial \theta}\hat{e}_\theta + \frac{\partial g}{\partial z}\hat{e}_z \tag{3}$$

5. Divergence of a vector $\vec{v}$:

$$\operatorname{div} \vec{v} \equiv \frac{1}{r}\frac{\partial \left(r v_r\right)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \tag{4}$$

6. Gradient of a vector $\vec{v}$:

$$\operatorname{grad} \vec{v} \equiv \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix} \tag{5}$$

7. Divergence of a tensor $\boldsymbol{S}$:

$$\begin{aligned} \operatorname{div} \boldsymbol{S} \equiv & \left[\frac{1}{r}\frac{\partial (r S_{rr})}{\partial r} + \frac{1}{r}\left(\frac{\partial S_{\theta r}}{\partial \theta} - S_{\theta\theta}\right) + \frac{\partial S_{zr}}{\partial z}\right]\hat{e}_r \\ & + \left[\frac{1}{r}\frac{\partial (r S_{r\theta})}{\partial r} + \frac{1}{r}\left(\frac{\partial S_{\theta\theta}}{\partial \theta} + S_{\theta r}\right) + \frac{\partial S_{z\theta}}{\partial z}\right]\hat{e}_\theta \\ & + \left[\frac{1}{r}\frac{\partial (r S_{rz})}{\partial r} + \frac{1}{r}\left(\frac{\partial S_{\theta z}}{\partial \theta}\right) + \frac{\partial S_{zz}}{\partial z}\right]\hat{e}_z \end{aligned} \tag{6}$$

8. "Convective" operator acting on a scalar, q:

$$(\vec{v} \cdot \operatorname{grad})q \equiv v_r \frac{\partial q}{\partial r} + \frac{v_\theta}{r}\frac{\partial q}{\partial \theta} + v_z \frac{\partial q}{\partial z} \tag{7}$$

[1] `http://tinyurl.com/6jodhw`

9. The usual "product" rule and divergence theorem

$$\operatorname{div}(g\vec{v}) = g(\operatorname{div}\vec{v}) + \operatorname{grad} g \cdot \vec{v} \tag{8}$$

$$\int_\Omega \operatorname{div}\vec{v}\,\mathrm{d}\Omega = \int_\Gamma \vec{v}\cdot\hat{n}\,\mathrm{d}\Gamma \tag{9}$$

still hold in the cylindrical coordinate system using the preceding definitions.

10. The "tensor" form of the product rule,

$$\operatorname{div}(\boldsymbol{S}\vec{v}) = \vec{v}\cdot\operatorname{div}\boldsymbol{S} + \boldsymbol{S}^T : \operatorname{grad}\vec{v} \tag{10}$$

also holds, where the "double-contraction" operator has the usual coordinate-system-independent definition, $\boldsymbol{A}:\boldsymbol{B} \equiv A_{ij}B_{ij}$. This identity is cumbersome to check for general $\vec{v}$, but we can e.g. verify it for the special case of $\vec{v} = (v_r, 0, 0)$. In this case, the second term on the right-hand side of (10) expands to:

$$\begin{aligned}\boldsymbol{S}^T : \operatorname{grad}\vec{v} &= \boldsymbol{S}^T : \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} & \frac{\partial v_r}{\partial z} \\ 0 & \frac{v_r}{r} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= S_{rr}\frac{\partial v_r}{\partial r} + \frac{S_{\theta r}}{r}\frac{\partial v_r}{\partial \theta} + S_{zr}\frac{\partial v_r}{\partial z} + S_{\theta\theta}\frac{v_r}{r} \end{aligned} \tag{11}$$

Meanwhile, the term on the left-hand side of (10) is:

$$\begin{aligned}\operatorname{div}(\boldsymbol{S}\vec{v}) &= \frac{1}{r}\frac{\partial(rS_{rr}v_r)}{\partial r} + \frac{1}{r}\frac{\partial(S_{\theta r}v_r)}{\partial\theta} + \frac{\partial(S_{zr}v_r)}{\partial z} \\ &= S_{rr}\frac{\partial v_r}{\partial r} + \frac{v_r}{r}\frac{\partial(rS_{rr})}{\partial r} + \frac{S_{\theta r}}{r}\frac{\partial v_r}{\partial\theta} + \frac{v_r}{r}\frac{\partial S_{\theta r}}{\partial\theta} + S_{zr}\frac{\partial v_r}{\partial z} + v_r\frac{\partial S_{zr}}{\partial z}\end{aligned} \tag{12}$$

Finally, taking (12)−(11) yields:

$$\operatorname{div}(\boldsymbol{S}\vec{v}) - \boldsymbol{S}^T : \operatorname{grad}\vec{v} = \frac{v_r}{r}\frac{\partial(rS_{rr})}{\partial r} + \frac{v_r}{r}\frac{\partial S_{\theta r}}{\partial\theta} + v_r\frac{\partial S_{zr}}{\partial z} - S_{\theta\theta}\frac{v_r}{r} \tag{13}$$

It is then easy to see that (13) $= \vec{v}\cdot\operatorname{div}\boldsymbol{S}$, for our special choice of $\vec{v}$, based on the definition of the divergence of a tensor given in (6). An analogous argument can be used to confirm the identity for $\vec{v} = (0, v_\theta, 0)$ and $\vec{v} = (0, 0, v_z)$, if desired. These three particular cases taken together then confirm (10).

2 Navier-Stokes Equations in Cylindrical Coordinates

In cylindrical coordinates, the Navier-Stokes equations are:

$$\rho\left(\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\operatorname{grad})\vec{u}\right) - \operatorname{div}\boldsymbol{\sigma} = \vec{f} \tag{14}$$

$$\operatorname{div}\vec{u} = 0 \tag{15}$$

where ρ is density, $\vec{u}$ is velocity, $\vec{f}$ is a body force, and $\boldsymbol{\sigma}$ is the total stress tensor, defined by:

$$\begin{aligned}
\boldsymbol{\sigma} &\equiv -p\boldsymbol{I} + \boldsymbol{\tau} \\
&= -p\boldsymbol{I} + \mu\left(\operatorname{grad}\vec{u} + (\operatorname{grad}\vec{u})^T\right) \\
&= -p\boldsymbol{I} + \mu \begin{bmatrix} 2\frac{\partial u_r}{\partial r} & \text{sym} & \text{sym} \\ r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta} & \frac{2}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) & \text{sym} \\ \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} & \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} & 2\frac{\partial u_z}{\partial z} \end{bmatrix}
\end{aligned} \tag{16}$$

where $\boldsymbol{I}$ is the identity matrix.

Consider a cylindrical domain Ω_c with radius R and height L. Multiplying Eqns. (14), (15) by test functions $(\vec{v}, q)$, integrating over Ω_c, and applying the divergence theorem yields the variational formulation of the problem, find $(\vec{u}, p)$ such that:

$$\begin{aligned}
&\int_{\Omega_c} \left[\rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\operatorname{grad})\vec{u}\right)\cdot\vec{v} - p\operatorname{div}\vec{v} + \boldsymbol{\tau} : \operatorname{grad}\vec{v} - \vec{f}\cdot\vec{v}\right] \mathrm{d}\Omega_c \\
&\qquad - \int_\Gamma (\boldsymbol{\sigma}\vec{v})\cdot\hat{n}\,\mathrm{d}\Gamma = 0
\end{aligned} \tag{17}$$

$$\int_{\Omega_c} q\operatorname{div}\vec{u}\,\mathrm{d}\Omega_c = 0 \tag{18}$$

holds for all admissible $(\vec{v}, q)$.

3 Axisymmetric Navier-Stokes Equations

Suppose that the velocity field is axisymmetric, that is:

$$u_\theta = \frac{\partial u_r}{\partial \theta} = \frac{\partial u_z}{\partial \theta} = 0 \tag{19}$$

and there is no forcing the the tangential direction ($f_\theta = 0$). Then, we can ignore the θ-component of (17), and "generate" the r and z-components of the momentum equation by selecting test functions $\vec{v} = (\psi, 0, 0)$ and $\vec{v} = (0, 0, \psi)$, respectively. The resulting component equations are then: find (u_r, u_z, p) such that

$$\begin{aligned}
&\int_{\Omega_c} \left\{\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z}\right)\psi - \frac{p}{r}\frac{\partial(r\psi)}{\partial r} + \tau_{rr}\frac{\partial \psi}{\partial r} + \frac{\tau_{\theta r}}{r}\frac{\partial \psi}{\partial \theta} + \tau_{zr}\frac{\partial \psi}{\partial z} + \tau_{\theta\theta}\frac{\psi}{r} - f_r\psi\right\}\mathrm{d}\Omega_c \\
&\qquad - \int_\Gamma \left(-p\hat{n}_r + \tau_{rr}\hat{n}_r + \tau_{\theta r}\hat{n}_\theta + \tau_{zr}\hat{n}_z\right)\psi\,\mathrm{d}\Gamma = 0
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\int_{\Omega_c} \left\{\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z}\right)\psi - p\frac{\partial \psi}{\partial z} + \tau_{zr}\frac{\partial \psi}{\partial r} + \frac{\tau_{z\theta}}{r}\frac{\partial \psi}{\partial \theta} + \tau_{zz}\frac{\partial \psi}{\partial z} - f_z\psi\right\}\mathrm{d}\Omega_c \\
&\qquad - \int_\Gamma \left(\tau_{rz}\hat{n}_r + \tau_{\theta z}\hat{n}_\theta - p\hat{n}_z + \tau_{zz}\hat{n}_z\right)\psi\,\mathrm{d}\Gamma = 0
\end{aligned} \tag{21}$$

holds for all admissible test functions (ψ, q). In the axisymmetric case, the viscous stress tensor from (16) reduces to:

$$\boldsymbol{\tau} = \begin{bmatrix} 2\frac{\partial u_r}{\partial r} & \text{sym} & \text{sym} \\ 0 & \frac{2u_r}{r} & \text{sym} \\ \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} & 0 & 2\frac{\partial u_z}{\partial z} \end{bmatrix} \tag{22}$$

Then, (20)–(21) can be further simplified and combined with the axi-symmetric mass conservation equation to give:

$$\int_{\Omega_c} \left\{ \rho \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} \right) \psi - p \frac{\partial \psi}{\partial r} - p\frac{\psi}{r} + \tau_{rr} \frac{\partial \psi}{\partial r} + \tau_{zr} \frac{\partial \psi}{\partial z} + \frac{2u_r \psi}{r^2} - f_r \psi \right\} \mathrm{d}\Omega_c - \int_{\Gamma} \left(-p\hat{n}_r + \tau_{rr}\hat{n}_r + \tau_{zr}\hat{n}_z \right) \psi \, \mathrm{d}\Gamma = 0 \tag{23}$$

$$\int_{\Omega_c} \left\{ \rho \left(\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} \right) \psi - p \frac{\partial \psi}{\partial z} + \tau_{zr} \frac{\partial \psi}{\partial r} + \tau_{zz} \frac{\partial \psi}{\partial z} - f_z \psi \right\} \mathrm{d}\Omega_c - \int_{\Gamma} \left(\tau_{rz}\hat{n}_r - p\hat{n}_z + \tau_{zz}\hat{n}_z \right) \psi \, \mathrm{d}\Gamma = 0 \tag{24}$$

$$\int_{\Omega_c} \left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \right) q \, \mathrm{d}\Omega_c = 0 \tag{25}$$

where the components of the simplified $\boldsymbol{\tau}$ matrix are given by (22), and we have also expanded the pressure term in (23). We can now see that (23)–(25) are equivalent to the Cartesian two-dimensional Navier-Stokes equations *plus* the highlighted terms, $-p\frac{\psi}{r}$, $\frac{2u_r\psi}{r^2}$, and $\frac{u_r}{r}$ in the r-momentum and mass conservation equations.